

Matrices commuting with a given normal tropical matrix

J. Linde
Dpto. de Algebra
Facultad de Matemáticas
Universidad Complutense
jorgelinde@ucm.es

M.J. de la Puente*
Dpto. de Algebra
Facultad de Matemáticas
Universidad Complutense
mpuente@mat.ucm.es
Phone: 34-91-3944659

AMS class.: 15A80; 14T05.

Keywords and phrases: tropical algebra, commuting matrices, normal matrix, idempotent matrix, alcoved polytope, convexity.

Abstract

Consider the space M_n^{nor} of square normal matrices $X = (x_{ij})$ over $\mathbb{R} \cup \{-\infty\}$, i.e., $-\infty \leq x_{ij} \leq 0$ and $x_{ii} = 0$. Endow M_n^{nor} with the tropical sum \oplus and multiplication \odot . Fix a real matrix $A \in M_n^{nor}$ and consider the set $\Omega(A)$ of matrices in M_n^{nor} which commute with A . We prove that $\Omega(A)$ is a finite union of alcoved polytopes; in particular, $\Omega(A)$ is a finite union of convex sets. The set $\Omega^A(A)$ of X such that $A \odot X = X \odot A = A$ is also a finite union of alcoved polytopes. The same is true for the set $\Omega'(A)$ of X such that $A \odot X = X \odot A = X$.

A topology is given to M_n^{nor} . Then, the set $\Omega^A(A)$ is a neighborhood of the identity matrix I . If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix. In one case, $\Omega(A)$ is a neighborhood of A . We give an upper bound for the dimension of $\Omega'(A)$. We explore the relationship between the polyhedral complexes $\text{span } A$, $\text{span } X$ and $\text{span}(AX)$, when A and X commute. Two matrices, denoted \underline{A} and \overline{A} , arise from A , in connection with $\Omega(A)$. The geometric meaning of them is given in detail, for one example. We produce examples of matrices which commute, in any dimension.

1 Introduction

Let $n \in \mathbb{N}$ and K be a field. Fix a matrix $A \in M_n(K)$ and consider $K[A]$, the algebra of polynomial expressions in A . In classical mathematics, the set $\Omega(A)$ of

*Corresponding author

matrices commuting with A is well-known: $\Omega(A)$ equals $K[A]$ if and only if the characteristic and minimal polynomials of A coincide. Otherwise, $K[A]$ is a proper linear subspace of $\Omega(A)$; see [27], chap. VII.

In this paper we study the analogous of $\Omega(A)$ in the tropical setting. Moreover, we restrict ourselves to square *normal* matrices over $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, i.e., matrices $A = (a_{ij})$ with $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j . The set of all such matrices, endowed with the tropical operations $\oplus = \max$ and $\odot = +$, is denoted M_n^{nor} .

For any $r \in \mathbb{R}_{\leq 0}$, the half-line $[-\infty, r) := \{x : -\infty \leq x < r\}$ is open in $\overline{\mathbb{R}}_{\leq 0}$ with the usual interval topology. A Cartesian product of such half-lines is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$ with the usual product topology. The half-line $(r, 0] := \{x : r < x \leq 0\}$ is open in $\overline{\mathbb{R}}_{\leq 0}$. A Cartesian product of such half-lines is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$.

The set M_n^{nor} can be identified with $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$ and, via this identification, M_n^{nor} gets a *topology*. Consider a matrix $X \in M_n^{nor}$ and a subset $V \subseteq M_n^{nor}$. We say that V is a *neighborhood* of X if there exists an open subset $U \subseteq M_n^{nor}$ such that $X \in U \subseteq V$ (we do not require V to be open).

Let $\Omega(A)$ be the subset of matrices commuting with a given real matrix A , i.e., $X \in M_n^{nor}$ such that $A \odot X = X \odot A$. The tropical analog of $K[A]$ inside M_n^{nor} is the set $\mathcal{P}(A)$ of tropical powers of A . In general, $\Omega(A)$ is larger than $\mathcal{P}(A)$ (see proposition 1).

Our new results are gathered in sections 3, 4 and 5. In section 3 we prove that

$$\Omega(A) = \bigcup_w \Omega_w(A)$$

is a finite union of alcoved polytopes, (see corollary 5). In particular, $\Omega(A)$ is a finite union of convex sets.

Two important subsets of $\Omega(A)$ are

$$\Omega^A(A) = \{X \in \Omega(A) : X \odot A = A \odot X = A\}$$

and

$$\Omega'(A) = \{X \in \Omega(A) : X \odot A = A \odot X = X\}.$$

Both are finite unions of alcoved polytopes (see theorems 9 and 12). Moreover, $\Omega^A(A)$ is a neighborhood (not necessarily open) of the identity matrix I . If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix 0 (see propositions 7 and 8).

The study of $\Omega^A(A)$ and $\Omega'(A)$ lead us to two matrices arising from A , denoted \underline{A} and \overline{A} , and we prove

$$\underline{A} \leq A \leq \overline{A},$$

(see proposition 17). Moreover, $X \leq \underline{A}$ is a necessary condition for $A \odot X = X \odot A = A$, and $\overline{A} \leq X$ is a necessary condition for $A \odot X = X \odot A = X$ (see corollary 15). This provides an upper bound for the dimension of $\Omega'(A)$ (see corollary 16). The matrix \underline{A} is explicitly given in expression (19), while the definition and computation of \overline{A} is more involved (see definition 14).

In section 4 we study some instances of commutativity of matrices under perturbations. Theorem 20 is an easy way to produce two real matrices in M_n^{nor} which commute. Another way to obtain two such matrices is given in theorem 22. The geometry is different in both instances: in the first case, the polyhedral complexes (i.e., tropical column spans) associated to the matrices are convex, but not so in the second. Under certain hypothesis we prove that $\Omega(A)$ is a neighborhood of A (see corollary 21).

Section 5 has an exploratory nature. We examine the relationship among the complexes $\text{span } A$, $\text{span } B$, $\text{span}(AB)$ and $\text{span}(BA)$ when commutativity is present or absent. In addition, the geometric meaning of the matrices \underline{A} , A and \overline{A} is given in full detail, for one example in the paper. We believe that classical convexity of $\text{span } A$ depends on the matrices \underline{A} and \overline{A} . We suspect that this is related to the question of commutativity. We leave two open questions in pages p. 15 and 20.

Alcoved polytopes play a crucial role in this paper. By definition, a polytope \mathcal{P} in \mathbb{R}^{n-1} is *alcoved* if it can be described by inequalities $c_i \leq x_i \leq b_i$ and $c_{ik} \leq x_i - x_k \leq b_{ik}$, for some $i, k \in [n-1]$, $i \neq k$, and $c_i, b_i, c_{ik}, b_{ik} \in \mathbb{R} \cup \{\pm\infty\}$. They are classically convex sets. Alcoved polytopes have been studied in [22, 23]. In connection with tropical mathematics, they appeared in [17, 18, 19, 29, 36]. *Kleene stars* are matrices A such that $A = A^*$, where $*$ is the so-called Kleene operator. Alcoved polytopes and Kleene stars are closely related notions; see [29, 32, 33].

By definition, a matrix $A = (a_{ij})$ over $\overline{\mathbb{R}}$ is *normal* if $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j . It is *strictly normal* if, in addition, $-\infty \leq a_{ij} < 0$, for all $i \neq j$. There are FOUR REASONS for us to restrict to normal matrices. First, it is not all too restrictive. Indeed, by the Hungarian Method (see [5, 6, 21, 26]), for every matrix A there exist a (not unique) similar matrix N which is normal. In practice, this means that by a relabeling of the columns of A and a translation, any A can be assumed to be normal. Second, normality of A has a clear geometric meaning in \mathbb{R}^{n-1} . Consider the alcoved polytope

$$C_A := \left\{ x \in \mathbb{R}^{n-1} : \begin{array}{l} a_{in} \leq x_i \leq -a_{ni} \\ a_{ik} \leq x_i - x_k \leq -a_{ki} \end{array} ; 1 \leq i \neq k \leq n-1 \right\}. \quad (1)$$

Then, A is normal if and only if the zero vector belongs to C_A and the columns of the matrix A_0 (see definition in p. 5), viewed as points in \mathbb{R}^{n-1} , lie around the zero vector and are listed in a predetermined order (and this order is a kind of orientation in \mathbb{R}^{n-1}); see [29] and also [14, 15, 16]. Third, when computing examples, normal

matrices are easy to handle, due to inequalities (2). Fourth and last, normal matrices satisfy many max-plus properties (e.g., they are strongly definite; see [6, 7]).

Some aspects of commutativity in tropical algebra (also called max-plus algebra or max-algebra) have been addressed earlier. It is known that two commuting matrices have a common eigenvector; see [7], sections 4.7, 5.3.5 and 9.2.2. In [20] it is proved that the critical digraphs of two commuting irreducible matrices have a common node.

2 Background and notations

For $n \in \mathbb{N}$, set $[n] := \{1, 2, \dots, n\}$. Let $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{\geq 0}$, $\overline{\mathbb{R}}_{\leq 0}$, etc. have the obvious meaning. On $\overline{\mathbb{R}}_{\leq 0}$, i.e., on the closed unbounded half-line $[-\infty, 0]$, we consider the *interval topology*: an open set in $[-\infty, 0]$ is either a finite intersection or an arbitrary union of sets of the form $[-\infty, a)$ or $(b, 0]$, with $-\infty < a, b < 0$.

$\oplus = \max$ is the tropical sum and $\odot = +$ is the tropical product. For instance, $3 \oplus (-7) = 3$ and $3 \odot (-7) = -4$. Define tropical sum and product of matrices following the same rules of classical linear algebra, but replacing addition (multiplication) by tropical addition (multiplication). Consider order n square matrices. The *tropical multiplicative identity* is $I = (\alpha_{ij})$, with $\alpha_{ii} = 0$ and $\alpha_{ij} = -\infty$, for $i \neq j$. The *zero matrix* is denoted 0 (every entry of it is null). We will never use classical multiplication of matrices; thus $A \odot X$ will be written AX , for matrices A, X , for simplicity.

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same order, then $A \leq B$ means $a_{ij} \leq b_{ij}$, for all i, j .

By definition, a square matrix $A = (a_{ij})$ over $\overline{\mathbb{R}}$ is *normal* if $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j . Thus, A is normal if and only if $I \leq A \leq 0$. Let us define A^0 to be the identity matrix I . So we have

$$I = A^0 \leq A \leq A^2 \leq A^3 \leq \dots \leq 0 \quad (2)$$

since tropical multiplication by any matrix is monotonic (because it amounts to computing certain sums and maxima). By a theorem of Yoeli's (see [37]), we have $A^{n-1} = A^n = A^{n+1} = \dots$ and we denote this matrix by A^* and call it the *Kleene star* of A . A matrix A is a Kleene star if $A = A^*$.

A normal matrix A is *strictly normal* if $a_{ij} < 0$, whenever $i \neq j$.

Let M_n^{nor} denote the family of order n normal matrices over $\overline{\mathbb{R}}$. It is in bijective correspondence with $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$. We consider the *product interval topology* on $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$. The bijection carries this topology onto M_n^{nor} . The *border* of M_n^{nor} is the set of matrices A such that $a_{ij} = 0$ or $-\infty$, for some $i \neq j$.

We will write the coordinates of points in \mathbb{R}^n in columns. Let $A \in \mathbb{R}^{n \times m}$ and denote by $a_1, \dots, a_m \in \mathbb{R}^n$ the columns of A . The (*tropical column*) *span* of A is, by definition,

$$\begin{aligned} \text{span } A : &= \{(\mu_1 \odot a_1) \oplus \dots \oplus (\mu_m \odot a_m) \in \mathbb{R}^n : \mu_1, \dots, \mu_m \in \mathbb{R}\} \\ &= \max\{\mu_1 u + a_1, \dots, \mu_m u + a_m : \mu_1, \dots, \mu_m \in \mathbb{R}\} \end{aligned} \quad (3)$$

where $u = (1, \dots, 1)^t$ and maxima are computed coordinatewise. We will never use classical linear spans in this paper. Clearly, the set $\text{span } A$ is closed under classical addition of the vector μu , for $\mu \in \mathbb{R}$, since $\odot = +$. Therefore, the hyperplane section $\{x_n = 0\} \cap \text{span } A$ determines $\text{span } A$ completely. The set $\{x_n = 0\} \cap \text{span } A$ is a connected polyhedral complex of impure dimension $\leq n - 1$ and it is not convex, in general. Let A be normal. Then $\text{span } A = C_A$ in (1) (and so it is convex) if and only if A is a Kleene-star; see [29, 32]. Throughout the paper, we will identify the hyperplane $\{x_n = 0\}$ inside \mathbb{R}^n with \mathbb{R}^{n-1} . In particular, columns of order n matrices having zero last row are considered as points in \mathbb{R}^{n-1} .

For any $d \in \mathbb{R}^n$, $\text{diag } d$ denotes the square matrix whose diagonal is d and is $-\infty$ elsewhere.

For any real matrix A , the matrix A_0 is defined as the tropical product

$$A \text{ diag}(-\text{row}(A, n)). \quad (4)$$

Thus, the j -th column of A_0 is a tropical multiple of the corresponding column of A (i.e., the j -th column of A_0 is the sum of the vector $-a_{nj}u$ and the j -th column of A). The last row of A_0 is zero. Therefore, the matrix A_0 is used to draw the complex $\{x_n = 0\} \cap \text{span } A$ inside \mathbb{R}^{n-1} . The sets $\text{span } A$ and $\{x_n = 0\} \cap \text{span } A$ determine each other.

The simplest objects in the tropical plane $\overline{\mathbb{R}}^2$ are lines. Given a tropical linear form

$$p_1 \odot X \oplus p_2 \odot Y \oplus p_3 = \max\{p_1 + X, p_2 + Y, p_3\}$$

a *tropical line* consists of the points $(x, y)^t$ where this *maximum is attained, at least, twice*. Such twice-attained-maximum condition is the tropical analog of the classical vanishing point set. Denote this line by L_p , where $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Lines in the tropical plane are *tripods*. Indeed, L_p is the union of three rays meeting at point $(p_3 - p_1, p_3 - p_2)^t$, in the directions west, south and north-east. The point is called the *vertex* of L_p .

Take $p = 0$. The line L_0 splits the plane $\overline{\mathbb{R}}^2$ into three closed sectors $S_1 := \{x \geq 0, x \geq y\}$, $S_2 := \{x \leq y, y \geq 0\}$ and $S_3 := \{x \leq 0, y \leq 0\}$. An order 3 real matrix A is normal if and only if (omitting the last row in A_0 , which is zero) each column of A_0 lies in the corresponding sector i.e., $\text{col}(A_0, j) \in S_j$, for $j = 1, 2, 3$. For instance, consider the normal matrix B and take B_0 in example 11, figure 3 top

centre, p. 20. Notice that $(5, 1)^t \in S_1$, $(-3, 0)^t \in S_2$ and $(-1, -6)^t \in S_3$. An analogous statement holds for $\overline{\mathbb{R}}^{n-1}$ and order n matrices. See [3, 4, 11, 12, 13, 24, 25, 30] for an introduction to tropical geometry. See [1, 2, 5, 7, 8, 9, 35, 38] for an introduction to tropical (or max-plus) algebra.

3 Normal matrices which commute with A

The set M_2^{nor} is commutative, since $AB = BA = A \oplus B$, for any $A, B \in M_2^{nor}$. Thus, we will study the set

$$\Omega(A) := \{X \in M_n^{nor} : AX = XA\}, \quad (5)$$

for a real matrix $A \in M_n^{nor}$ and $n \geq 3$.

If $A \in M_n^{nor}$ is real and $\lambda \in \mathbb{R}$, then $\lambda \odot A = \lambda u + A$ is normal if and only if $\lambda = 0$, where u denotes the order n one matrix. Together with (2), this means that the tropical analog of $K[A]$ inside M_n^{nor} is the set of powers of A together with the zero matrix

$$\mathcal{P}(A) := \{I = A^0, A, A^2, \dots, A^{n-1} = A^*, 0\}. \quad (6)$$

For $A \in M_n^{nor}$ real, set

$$m(A) := \min_{i,j \in [n]} a_{ij} = \min_{i \neq j \in [n]} a_{ij} \in \mathbb{R}_{\leq 0}, \quad M(A) := \max_{i \neq j \in [n]} a_{ij} \in \mathbb{R}_{\leq 0}. \quad (7)$$

For each $r \in \mathbb{R}$, and $i, j \in [n]$, $i \neq j$, let $E_{ij}(r) \in M_n^{nor}$ denote the matrix whose (i, j) entry equals r , being zero everywhere else. For a generic $A \in M_n^{nor}$ the matrix $E_{ij}(r)$ is not a power of A .

The following proposition shows that, in general, $\Omega(A)$ is larger than $\mathcal{P}(A)$.

Proposition 1. *For any real $A \in M_n^{nor}$ there exist $\epsilon > 0$ and $i, j \in [n]$ with $i \neq j$ such that $E_{ij}(-\epsilon) \in \Omega(A)$.*

Proof. Fix i, j and ϵ . We have $AE_{ij}(-\epsilon) = E_{ij}(\alpha)$ and $E_{ij}(-\epsilon)A = E_{ij}(\beta)$, where

$$\alpha = \max\{a_{i1}, \dots, a_{i,i-1}, -\epsilon, a_{i,i+1}, \dots, a_{in}\} \text{ and}$$

$$\beta = \max\{a_{1j}, \dots, a_{j-1,j}, -\epsilon, a_{j+1,j}, \dots, a_{nj}\}.$$

If $a_{ij} = 0$, then $\alpha = \beta = a_{ij} = 0$, whence $AE_{ij}(-\epsilon) = E_{ij}(-\epsilon)A = 0$.

Assume now that A is strictly normal. Then $M(A) < 0$. For any ϵ with $M(A) < -\epsilon < 0$ and any $i \neq j$, we have $\alpha = \beta = -\epsilon$, whence $AE_{ij}(-\epsilon) = E_{ij}(-\epsilon)A = E_{ij}(-\epsilon)$. \square

Let W_n be the set of empty-diagonal order n matrices with entries in $[n]^2$ (the diagonal is irrelevant in these matrices). Each $w \in W_n$ is called a *winning position* or a *winner*. Set

$$\begin{aligned} \Omega_w(A) := \{X \in \Omega(A) : (AX)_{ij} &= a_{i,w(i,j)_1} + x_{w(i,j)_1,j} = \\ (XA)_{ij} &= x_{i,w(i,j)_2} + a_{w(i,j)_2,j}, \text{ for } i, j \in [n], i \neq j\}. \end{aligned} \quad (8)$$

Example 2. Consider

$$A = \begin{bmatrix} 0 & -4 & -6 & -3 \\ -6 & 0 & -4 & -3 \\ -3 & -6 & 0 & -3 \\ -6 & -3 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 & -4 & -6 \\ -2 & 0 & -3 & -4 \\ -5 & -6 & 0 & -5 \\ -6 & -5 & -2 & 0 \end{bmatrix}.$$

Then

$$AB = BA = \begin{bmatrix} 0 & -4 & -4 & -3 \\ -2 & 0 & -3 & -3 \\ -3 & -6 & 0 & -3 \\ -5 & -3 & -2 & 0 \end{bmatrix}$$

so that $B \in \Omega_w(A)$ with

$$w = \begin{bmatrix} (1,1) & (1,3) & (4,1) \\ (2,1) & (2,3) & (4,2) \\ (1,3) & (2,2) & (4,3) \\ (2,3) & (2,4) & (4,3) \end{bmatrix}. \quad \square$$

Example 3. For any real $A \in M_n^{nor}$,

- if tr denotes the transposition operator, then $I \in \Omega_{\text{tr}}(A)$,
- if id denotes the identity operator, then $0, A^* \in \Omega_{\text{id}}(A)$.

Proposition 4. For any real $A \in M_n^{nor}$, $\Omega_w(A)$ is an alcoved polytope.

Proof. Fix $i, j \in [n], i \neq j$. Then (8) means that

$$a_{i,w(i,j)_1} + x_{w(i,j)_1,j} = x_{i,w(i,j)_2} + a_{w(i,j)_2,j} \quad (9)$$

and the following $2n - 2$ inequalities hold

$$a_{is} + x_{sj} \leq a_{i,w(i,j)_1} + x_{w(i,j)_1,j}, \text{ for } s \neq w(i,j)_1, \quad (10)$$

$$x_{it} + a_{tj} \leq x_{i,w(i,j)_2} + a_{w(i,j)_2,j}, \text{ for } t \neq w(i,j)_2. \quad (11)$$

Equalities and inequalities (9), (10) and (11) show that $X \in \Omega_w(A)$ if and only if $X = (x_{ij})$ belongs to certain alcoved polytope in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n} \simeq M_n^{nor}$. \square

Remark 1: Given a winner w , if there exist $i, j, s, t \in [n]$ with $i \neq j$ and $s \neq t$ such that

$$(i, j) \neq (s, t) \neq (j, i), \quad w(i, j) = (s, t), \quad w(s, t) = (i, j), \quad a_{is} + a_{si} \neq a_{jt} + a_{tj}, \quad (12)$$

then $\Omega_w(A)$ is empty. Indeed, by (9), the following two parallel hyperplanes

$$a_{is} + x_{sj} = x_{it} + a_{tj}, \quad a_{si} + x_{it} = x_{sj} + a_{jt},$$

take part in the description of $\Omega_w(A)$.

For instance, back to A in example 2, if $\tau \in W_n$ is such that $\tau(1, 3) = (2, 4)$ and $\tau(2, 4) = (1, 3)$, then $\Omega_\tau(A) = \emptyset$, because $a_{12} + a_{21} = -10 \neq a_{34} + a_{43} = -6$.

Remark 2: Given a winner w and $i, j \in [n], i \neq j$, if

$$w(i, j) = (i, j) \text{ or } w(i, j) = (j, i), \quad (13)$$

then equality (9) is tautological. In particular,

$$\dim \Omega_w(A) \leq n^2 - n - \text{card } P_w^c, \quad (14)$$

where $P_w := \{(i, j) : 1 \leq i < j \leq n \text{ with } w(i, j) = (i, j) \text{ or } w(i, j) = (j, i)\}$ and c denotes complementary.

Example 2. (Continued) For w , the pairs which do not satisfy (13) are $w(1, 2) = (1, 1)$, $w(3, 2) = (2, 2)$ and $w(4, 1) = (2, 3)$, so that $P_w^c = \{(1, 2), (3, 2), (4, 1)\}$. It follows that $x_{12} = -4$, $x_{32} = -6$ and $x_{21} = x_{43}$ are some of the equations describing $\Omega_w(A)$. Besides, condition (12) is satisfied for no pairs, whence

$$0 < \dim \Omega_w(A) \leq 16 - 4 - 3 = 9. \quad \square$$

Clearly,

$$\Omega(A) = \bigcup_{w \in W_n} \Omega_w(A) \quad (15)$$

and the set W_n is finite, whence the following corollary is a straightforward consequence of proposition 4.

Corollary 5. For any real $A \in M_n^{nor}$, $\Omega(A)$ is a finite union of alcoved polytopes. \square

The sets $\Omega_w(A)$ are not too natural. On the contrary, the sets $\Omega^S(A)$ described below are more natural but harder to study. For any $S \in M_n^{nor}$, let

$$\Omega^S(A) := \{X \in \Omega(A) : XA = AX = S\}, \quad (16)$$

so that

$$\Omega(A) = \bigcup_{S \in M_n^{nor}} \Omega^S(A) \quad (17)$$

is a disjoint union. For instance, $B \in \Omega^S(A)$, for $S := BA$ in example 2. We also consider the set

$$\Omega'(A) := \{X \in \Omega(A) : XA = AX = X\}. \quad (18)$$

It is immediate to see that

1. $A^{j-1} \in \Omega^{A^j}(A)$, for $j \in [n]$. In particular, $I = A^0 \in \Omega^A(A)$, i.e., $AI = IA = A$.
2. $A^* \in \Omega'(A)$, i.e., $AA^* = A^*A = A^*$.
3. $0 \in \Omega'(A)$, i.e., $A0 = 0A = 0$.

Proposition 6. *For any real $A, B \in M_n^{nor}$, if that $A^{n-2} \leq B \leq A^*$, then $B \in \Omega^{A^*}(A)$.*

Proof. $A^{n-1} = A^n = A^{n+1} = \dots = A^*$, by Yoeli's theorem, and left or right multiplication by A is monotonic, so that $A^{n-2} \leq B \leq A^*$ implies $A^* \leq AB \leq A^*$ and $A^* \leq BA \leq A^*$. \square

Recall $m(A)$ and $M(A)$ defined in (7). Recall the topology in M_n^{nor} , defined in p. 2.

For $r \in \overline{\mathbb{R}}$, denote by $K(r) = (\alpha_{ij})$ the constant matrix such that $\alpha_{ii} = 0$ and $\alpha_{ij} = r$, for all $i \neq j$. For instance, $I = K(-\infty)$ and $0 = K(0)$.

Proposition 7. *For any real $A \in M_n^{nor}$, if $I \leq B \leq K(m(A))$, then $B \in \Omega^A(A)$. In particular, $\Omega^A(A)$ is a neighborhood of the identity matrix I .*

Proof. The hypothesis $I \leq B \leq K(m(A))$ means that B is normal and $b_{ij} \leq m(A)$, for all $i \neq j$.

If $i \neq j$, we have $(AB)_{ij} = \max_{k \in [n]} a_{ik} + b_{kj} = a_{ij}$, since $a_{ik} + b_{kj} \leq a_{ik} + m(A) \leq m(A) \leq a_{ij}$, when $k \neq j$, and $a_{ij} + b_{jj} = a_{ij}$. Similarly, $(BA)_{ij} = a_{ij}$. This shows $AB = BA = A$, so that $B \in \Omega^A(A)$.

The value $m(A)$ defined in (7) is real. The set $U = \{B : I \leq B < K(m(A))\}$ is in bijective correspondence with the Cartesian product of half-lines $[-\infty, m(A))^{n^2-n}$, which is open. Moreover, $I \in U \subseteq \Omega^A(A)$, proving the neighborhood condition. \square

Notice that $m(A)$ equals $-|||A|||$, as defined in [29]. There, it is proved that $|||A|||$ is the (tropical) radius of the section $\{x_n = 0\} \cap \text{span } A$, i.e., the maximal tropical distance to the zero vector, from any point on $\{x_n = 0\} \cap \text{span } A$. This conveys a geometrical meaning to proposition 7.

Proposition 8. Suppose that $A \in M_n^{nor}$ is real and strictly normal. If B is such that $K(M(A)) \leq B \leq 0$, then $B \in \Omega'(A)$. In particular, $\Omega'(A)$ is a neighborhood of the zero matrix 0.

Proof. We have $M(A) < 0$, by strict normality. The hypothesis on $B = (b_{ij})$ means that $M(A) \leq b_{ij}$, for every $i, j \in [n]$ with $i \neq j$.

For $i \neq j$, we get $(AB)_{ij} = \max_{k \in [n]} a_{ik} + b_{kj} = b_{ij}$, since $a_{ik} + b_{kj} \leq M(A) + b_{kj} \leq M(A) \leq b_{ij}$, when $k \neq i$, and $a_{ii} + b_{ij} = b_{ij}$. Similarly, $(BA)_{ij} = b_{ij}$. This shows $AB = BA = B$, so that $B \in \Omega'(A)$.

The set $U = \{B : K(M(A)) < B \leq 0\}$ is in bijective correspondence with the Cartesian product of half-lines $(M(A), 0]^{n^2-n}$, which is open. Moreover, $0 \in U \subseteq \Omega'(A)$, proving the neighborhood condition. \square

Note that the former proposition is analogous to proposition 7, with the zero matrix playing the role of the identity matrix.

Below we describe the sets $\Omega^A(A)$ and $\Omega'(A)$ as finite union of alcoved polytopes. In order to do so, for $i \in [n]$, consider the matrices

- $R_A^i = (r_{kj}^i)$, with $r_{kj}^i = a_{ij} - a_{ik}$ (difference in i -th row; subscripts k, j get inverted),
- $C_A^i = (c_{kj}^i)$, with $c_{kj}^i = a_{ki} - a_{ji}$ (difference in i -th column; subscripts k, j don't get inverted).

Let \oplus' denote min. Write $R := \bigoplus'_{i \in [n]} R_A^i$ and $C := \bigoplus'_{i \in [n]} C_A^i$ and consider

$$\underline{A} := R \oplus' C = A \oplus' R \oplus' C, \quad (19)$$

the last equality being true since $r_{ij}^i = a_{ij}$ and $c_{kj}^j = a_{kj}$, by normality of A . Clearly, $\underline{A} \leq A$ and \underline{A} is real and normal, if A is.

Notation: $(\leftarrow, \underline{A}) := \{X \in M_n^{nor} : X \leq \underline{A}\}$. This is an alcoved polytope of dimension $n^2 - n$.

Theorem 9. For any real $A \in M_n^{nor}$, $\Omega^A(A)$ is a finite union of alcoved polytopes. Moreover,

$$\Omega_{\text{tr}}(A) \subseteq \Omega^A(A) \subseteq (\leftarrow, \underline{A}).$$

Proof. $AX = XA = A$ if and only if

$$\max_{k \in [n]} a_{ik} + x_{kj} = a_{ij}, \quad \max_{k \in [n]} x_{ik} + a_{kj} = a_{ij}, \quad \text{for } i, j \in [n], i \neq j. \quad (20)$$

Now, for each $X = (x_{ij}) \in \Omega^A(A)$ there exists some winner w_X such that, for each pair (i, j) with $i \neq j$, the maxima in (20) are attained at $w_X(i, j)$. Since W_n is finite, then (20) describe a finite union of alcoved polytopes in the variables x_{ij} . Moreover, $X \leq \underline{A}$ follows from (19) and (20). In addition, the maxima in (20) are attained, at least, for the transposition operator. Therefore, $\Omega_{\text{tr}}(A) \subseteq \Omega^A(A)$. \square

Algorithm 10. To compute \underline{A} , we proceed as follows: for $1 \leq i < j \leq n$,

- compute the minimum and maximum of $\text{row}(A, i) - \text{row}(A, j)$, denoted mr_{ij} and MR_{ij} , respectively,
- compute the minimum and maximum of $\text{col}(A, i) - \text{col}(A, j)$, denoted mc_{ij} and MC_{ij} , respectively,
- $\underline{A}_{ij} = \min\{a_{ij}, \text{mr}_{ij}, -\text{MC}_{ij}\}$,
- $\underline{A}_{ji} = \min\{a_{ji}, -\text{MR}_{ij}, \text{mc}_{ij}\}$.

A sorting algorithm is needed to compute $\text{mr}_{ij}, \text{mc}_{ij}, \text{MR}_{ij}, \text{MC}_{ij}$. For instance, Mergesort has $O(n \log n)$ complexity, whence the complexity of the computation of \underline{A} is $O(n^3 \log n)$.

Example 11. For

$$B = \begin{bmatrix} 0 & -3 & -1 \\ -4 & 0 & -6 \\ -5 & 0 & 0 \end{bmatrix} \text{ we get } \underline{B} = \begin{bmatrix} 0 & -3 & -3 \\ -5 & 0 & -6 \\ -5 & -2 & 0 \end{bmatrix}. \quad (21)$$

On the other hand, for A in example 2, we get $A = \underline{A}$. \square

Notation: $[A, \rightarrow) := \{X \in M_n^{\text{nor}} : A \leq \underline{X}\}$. It is an alcoved polytope, since the definition of \underline{X} involves differences $x_{ij} - x_{kl}$ of two entries.

The proof of the theorem below is similar to the proof of theorem 9. Alternatively, theorem 12 is a corollary of theorem 9, using that $X \in \Omega^A(A)$ if and only if $A \in \Omega'(X)$.

Theorem 12. For any real $A \in M_n^{\text{nor}}$, $\Omega'(A)$ is a finite union of alcoved polytopes. Moreover,

$$\Omega_{\text{id}}(A) \subseteq \Omega'(A) \subseteq [A, \rightarrow). \quad \square$$

The sets $(\leftarrow, \underline{A}]$ and $[\underline{A}, \rightarrow)$ are alcoved polytopes, but $[\underline{A}, \rightarrow)$ is trickier than $(\leftarrow, \underline{A}]$. We can compute a *tight description* of any of them, as explained in [29]. It goes as follows. For any $m \in \mathbb{N}$, any real matrix $H \in M_m^{\text{nor}}$ yields the alcoved polytope C_H (see (1)), and it turns out that $C_H = C_{H^*}$. Moreover, the description of this convex set given by H^* is *tight*.

Example 11. (Continued) Let us compute a tight description of $[B, \rightarrow)$, for B in (21). The matrix \underline{X} is defined in (19) and we have $B \leq \underline{X}$ if and only if

$$\begin{array}{ll}
-3 \leq x_{12} & -6 \leq x_{23} \\
-3 \leq x_{32} - x_{31} & -6 \leq x_{13} - x_{12} \\
-3 \leq x_{13} - x_{23} & -6 \leq x_{21} - x_{31} \\
-1 \leq x_{13} & -5 \leq x_{31} \\
-1 \leq x_{23} - x_{21} & -5 \leq x_{21} - x_{23} \\
-1 \leq x_{12} - x_{32} & -5 \leq x_{32} - x_{12} \\
-4 \leq x_{21} & 0 \leq x_{32} \\
-4 \leq x_{31} - x_{32} & 0 \leq x_{12} - x_{13} \\
-4 \leq x_{23} - x_{13} & 0 \leq x_{31} - x_{21}.
\end{array}$$

Now, in order to write down the matrix H , we perform a relabeling of the unknowns; for instance:

$$y_1 = x_{12}, y_2 = x_{13}, y_3 = x_{21}, y_4 = x_{23}, y_5 = x_{31}, y_6 = x_{32},$$

so that,

$$\begin{array}{ll}
-3 \leq y_1 & 0 \leq y_1 - y_2 \leq 6 \\
-1 \leq y_2 & -1 \leq y_1 - y_6 \leq 5 \\
-4 \leq y_3 & -3 \leq y_2 - y_4 \leq 4 \\
-6 \leq y_4 & -5 \leq y_3 - y_4 \leq 1 \\
-5 \leq y_5 & -6 \leq y_3 - y_5 \leq 0 \\
0 \leq y_6 & -4 \leq y_5 - y_6 \leq 3
\end{array}$$

and we get $\underline{[B, \rightarrow)} = C_H$, with

$$H = \begin{bmatrix} 0 & 0 & -\infty & -\infty & -\infty & -1 & -3 \\ -6 & 0 & -\infty & -3 & -\infty & -\infty & -1 \\ -\infty & -\infty & 0 & -5 & -6 & -\infty & -4 \\ -\infty & -4 & -1 & 0 & -\infty & -\infty & -6 \\ -\infty & -\infty & 0 & -\infty & 0 & -4 & -5 \\ -5 & -\infty & -\infty & -\infty & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $H^3 = H^4 = H^*$, with

$$H^* = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -4 & -4 & 0 & -4 & -4 & -4 & -4 \\ -5 & -4 & -1 & 0 & -5 & -5 & -5 \\ -4 & -4 & 0 & -4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\underline{[B, \rightarrow]} = C_H = C_{H^*}$, by [29], and this set is described tightly as follows:

$$\begin{array}{ll} -1 \leq y_1 \leq 0 & -1 \leq y_1 - y_4 \leq 5 \\ -1 \leq y_2 \leq 0 & -1 \leq y_1 - y_5 \leq 4 \\ -4 \leq y_3 \leq 0 & -1 \leq y_2 - y_3 \leq 4 \\ -5 \leq y_4 \leq 0 & -1 \leq y_2 - y_4 \leq 4 \\ -4 \leq y_5 \leq 0 & -1 \leq y_2 - y_5 \leq 4 \\ 0 = y_6 & -4 \leq y_3 - y_4 \leq 1 \\ 0 \leq y_1 - y_2 \leq 1 & -4 \leq y_3 - y_5 \leq 0 \\ -1 \leq y_1 - y_3 \leq 4 & -5 \leq y_4 - y_5 \leq 4. \end{array}$$

In particular, $\dim \underline{[B, \rightarrow]} = \dim C_{H^*} = 9 - 3 - 1 = 5$. Undoing the relabeling, we get

$$\begin{array}{ll} -1 \leq x_{12} \leq 0 & -1 \leq x_{12} - x_{23} \leq 5 \\ -1 \leq x_{13} \leq 0 & -1 \leq x_{12} - x_{31} \leq 4 \\ -4 \leq x_{21} \leq 0 & -1 \leq x_{13} - x_{21} \leq 4 \\ -5 \leq x_{23} \leq 0 & -1 \leq x_{13} - x_{23} \leq 4 \\ -4 \leq x_{31} \leq 0 & -1 \leq x_{13} - x_{31} \leq 4 \\ 0 = x_{32} & -4 \leq x_{21} - x_{23} \leq 1 \\ 0 \leq x_{12} - x_{13} \leq 1 & -4 \leq x_{21} - x_{31} \leq 0 \\ -1 \leq x_{12} - x_{21} \leq 4 & -5 \leq x_{23} - x_{31} \leq 4. \end{array}$$

Write

$$\overline{B} = \begin{bmatrix} 0 & -1 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix} \quad (22)$$

and notice that $\overline{B} \leq X$ follows from the first six inequalities above.

Computations as in the former example can be done for any real matrix $A \in M_n^{nor}$, as follows.

Definition 13. For $n \in \mathbb{N}$, a relabeling is a bijection between two sets of variables: $\{x_{ij} : (i, j) \in [n]^2, i \neq j\}$ and $\{y_k : k \in [n^2 - n]\}$. By abuse of notation, we write $y_k = x_{ij}$, for corresponding y_k and x_{ij} .

Definition 14. Given $A \in M_n^{nor}$ real, suppose that $[A, \rightarrow)$ equals C_{H^*} , for some idempotent matrix $H^* = (h_{ij}^*) \in M_{n^2-n+1}^{nor}$ and some relabeling $y_k = x_{ij}$. Then $\bar{A} = (\alpha_{ij}) \in M_n^{nor}$, with $\alpha_{ij} = h_{k, n^2-n+1}^*$, i.e., the entries of \bar{A} are obtained from the last column of H^* .

The matrix \bar{A} does not depend on the relabeling. The arithmetical complexity of computing \bar{A} is that of H^* , which is $O((n^2 - n)^3) = O(n^6)$, by the Floyd–Warshall algorithm.

Corollary 15. For any $A, X \in M_n^{nor}$ with A real, $A \leq \underline{X}$ implies $\bar{A} \leq X$. In particular, $\Omega'(A) \subseteq [\bar{A}, \rightarrow)$.

Proof. We proceed as in example above and we use theorem 12. \square

Corollary 16. Given $A \in M_n^{nor}$ real, suppose that $[A, \rightarrow)$ equals C_{H^*} , for some idempotent matrix $H^* = (h_{ij}^*) \in M_{n^2-n+1}^{nor}$. Then

$$\dim \Omega'(A) \leq n^2 - n - \text{card } Q,$$

where $Q = \{(i, n^2 - n + 1) : h_{i, n^2-n+1}^* = h_{n^2-n+1, i}^* = 0, \text{ with } 1 \leq i < n^2 - n + 1\} \cup \{(i, k) : h_{ik}^* = h_{ki}^* = 0, \text{ with } 1 \leq i < k \leq n^2 - n + 1\}$.

Proof. The description of $[A, \rightarrow)$ via H^* is tight, by proposition 2.6 in [29]. Thus, the dimension of $[A, \rightarrow)$ drops by one unit each time that a chain of two inequalities in expression (1) (for \bar{H}^* instead of A), turns into two equalities, which occurs whenever $h_{ik}^* = h_{ki}^* = 0$, by normality of H^* . Thus, $\dim [A, \rightarrow) = n^2 - n - \text{card } Q$ and this is an upper bound for $\dim \Omega'(A)$. \square

Proposition 17. For any $A \in M_n^{nor}$ real, we have $\underline{A} \leq A \leq \bar{A}$.

Proof. The inequality $\underline{A} \leq A$ was explained in p. 10. Now consider X such that $A \leq \underline{X}$. Then,

$$A \leq \underline{X} \leq X,$$

by the same reason, so that $A \leq X$. By definition 14, the matrix \bar{A} is obtained from the last column of H^* and, by [29], the description of the alcoved polytope $[A, \rightarrow)$ as C_{H^*} is tight. Part of this description is $\bar{A} \leq X$. Therefore, $A \leq \bar{A} \leq X$, by tightness. \square

Some questions arise, such as:

1. We know that $\underline{A} \leq A \leq \overline{A}$. Does every X with $\underline{A} \leq X \leq \overline{A}$ commute with A ? The answer is NO. Example: take B in (21) and

$$X = \begin{bmatrix} 0 & -2 & -2 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}, BX = \overline{B} \neq XB = \begin{bmatrix} 0 & -2 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}.$$

2. We know that A^* and 0 belong to $\Omega'(A)$. Does every X with $A^* \leq X \leq 0$ commute with A ? The answer is NO. Example: for B in (21), we have $B^* = \overline{B}$ in (22) and

$$X = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = XB \neq BX = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

4 Perturbations

Definition 18. Assume $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b$. Then a, b are of the same size if $b \leq 2a$. Otherwise, $2a < b$ and we say that a is small with respect to b .

In the topological space $M_n^{nor} \simeq \overline{\mathbb{R}}_{\leq 0}^{n^2-n}$ the following is expected to hold true, for any real matrix $A \in M_n^{nor}$:

1. for $j \in [n]$ and each *sufficiently small perturbation* X of A^{j-1} , we have $AX = XA$, and this is a perturbation of A^j , (including the case that X is a perturbation of $I = A^0$ or of $A^* = A^{n-1}$)
2. for each *sufficiently small perturbation* X of 0, we have $AX = XA$, and this is a perturbation of 0.

The point here is, of course, to give a precise meaning of *sufficiently small perturbation*. Although we are not able to do it yet, we believe that the statement will be about linear inequalities in terms of the non-zero entries a_{ij} of A and some perturbing constants $\pm\epsilon_1, \dots, \pm\epsilon_s$, with $\epsilon_k \geq 0$ for $k = 1, \dots, s$, and some $s \geq 0$. We further believe that the perturbing constants must be *small* with respect to every non-zero absolute value $|a_{ij}|$, according to definition 18. Recall that $\Omega(A)$ is larger than $\mathcal{P}(A)$ (see p. 2). An intriguing related QUESTION is the following: is every $X \in \Omega(A)$ a *small perturbation* of some member of $\mathcal{P}(A)$?

Below we present some partial results.

For brevity, write $A \oplus B := M = (m_{ij})$.

Proposition 19. Assume $A, B \in M_n^{nor}$ are such that $a_{ik} + b_{kj} \leq m_{ij}$, for all $i, j, k \in [n]$. Then $AB = BA = M$. In particular, $B \in \Omega^M(A)$.

Proof. By normality, $I \leq A \leq 0$ and $I \leq B \leq 0$, whence $A \leq AB \leq 0$ and $B \leq AB \leq 0$, since (tropical) left or right multiplication by any matrix is monotonic. Thus, $M \leq AB$ and, similarly, $M \leq BA$ and, by hypothesis, $AB \leq M$ and $BA \leq M$. Therefore $AB = BA = M$. \square

Theorem 20. *For each $n \in \mathbb{N}$ and each non positive real number r , any two order n matrices A, B having zero diagonal and all off-diagonal entries in the closed interval $[2r, r]$ satisfy $AB = BA = M$. In particular, $B \in \Omega^M(A)$.*

Proof. Let $a_{ii} = b_{ii} = 0$ and $2r \leq a_{ij}, b_{ij} \leq r \leq 0$, for $i, j \in [n]$. Fix $i, j \in [n]$ with $i \neq j$. For each $k \in [n]$, we have $a_{ik} + b_{kj} \leq 2r \leq a_{ij}, b_{ij}$, and we can apply the previous proposition to conclude. \square

That is an easy way to produce two real matrices which commute! Moreover, the matrices A, B and M are idempotent. Indeed, $A \leq A^2$ by normality and, since $a_{ij} + a_{jk} \leq 2r \leq a_{ik}$, we get $A^2 \leq A$, whence $A = A^2$; similarly $B = B^2$ and $M = M^2$. Here $B \in \Omega(A)$ is a perturbation of A and $AB = BA = M$ is a perturbation of $A^2 = A$, so this is an example of item 1 in p. 15, for $j = 2$.

In the former theorem, notice that the absolute value of the entries $|a_{ij}|$ and $|b_{ij}|$ of A and B are of the *same size*, taken by pairs, as in definition 18. The reader should compare theorem 20 with example 2, where $M^2 = AB = BA \neq M$, these matrices being different only at entry $(4, 1)$. There A, B and AB are idempotent, but M is not.

Corollary 21. *For each $n \in \mathbb{N}$ and each negative real number r , take a_{ij} in the open interval $(2r, r)$, whenever $i \neq j$ and $a_{ii} = 0$, all $i, j \in [n]$. Then $A = (a_{ij})$ is strictly normal and $\Omega(A)$ is a neighborhood of A .*

Proof. The Cartesian product of intervals $U = (2r, r)^{n^2-n}$ is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$. The image U' of U in M_n^{nor} satisfies $A \in U' \subseteq \Omega(A)$, by theorem 20, proving the neighborhood condition. \square

Corollary 21 is an instance of item 1 in p. 15. Below we present another one.

For $n \geq 3$, consider $p = (p_1, \dots, p_n) \in \mathbb{R}_{\geq 0}^n$ and $\epsilon \geq 0$ and set

$$P(-p, -\epsilon) := \begin{bmatrix} 0 & -\epsilon & \cdots & -\epsilon & -p_n \\ -p_1 & 0 & -\epsilon & \cdots & -\epsilon \\ -\epsilon & -p_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\epsilon \\ -\epsilon & \cdots & -\epsilon & -p_{n-1} & 0 \end{bmatrix} \in M_n^{nor}, \quad (23)$$

and for $n \geq 4$, set

$$Q(-p, -\epsilon) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\epsilon & -p_n \\ -p_1 & 0 & \cdots & \cdots & 0 & -\epsilon \\ -\epsilon & -p_2 & 0 & \cdots & \cdots & 0 \\ 0 & -\epsilon & -p_3 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\epsilon & -p_{n-1} & 0 \end{bmatrix} \in M_n^{nor}. \quad (24)$$

The matrices $P(-p, -\epsilon)$ and $Q(-p, -\epsilon)$ are perturbations of $P(-p, 0) = Q(-p, 0)$.

Theorem 22. Let $p \in \mathbb{R}_{\geq 0}^n$ and let $\delta, \epsilon \geq 0$ be such that $\delta + \epsilon \leq \min_{i \in [n]} p_i$. Write $m = \min\{\delta, \epsilon\}$. Then

1. $P(-p, -\delta)P(-p, -\epsilon) = P(-p, -\epsilon)P(-p, -\delta) = P(-(\delta + \epsilon, \dots, \delta + \epsilon), -m)$.
2. $Q(-p, -\delta)Q(-p, -\epsilon) = Q(-p, -\epsilon)Q(-p, -\delta) = Q(-(m, \dots, m), 0)$.

Proof. Straightforward computations. \square

Example 23. Take $p = (4, 3, 5)$, $\epsilon = 1$ and $\delta = 2$,

$$P(-p, -2) = \begin{bmatrix} 0 & -2 & -5 \\ -4 & 0 & -2 \\ -2 & -3 & 0 \end{bmatrix}, \quad P(-p, -1) = \begin{bmatrix} 0 & -1 & -5 \\ -4 & 0 & -1 \\ -1 & -3 & 0 \end{bmatrix}. \quad (25)$$

By theorem 22, we have

$$P(-p, -2)P(-p, -1) = P(-p, -1)P(-p, -2) = P(-(3, 3, 3), -1) = \begin{bmatrix} 0 & -1 & -3 \\ -3 & 0 & -1 \\ -1 & -3 & 0 \end{bmatrix}. \quad (26)$$

Pictures for this example are shown in figure 1. Write $A = P(-p, -2)$, $B = P(-p, -1)$, $C = AB = BA$. In \mathbb{R}^2 we have sketched the intersection of the classical hyperplane $\{x_3 = 0\}$ with $\text{span } A, \text{span } P(-p, 0)$ and $\text{span } B$ on top, and with $\text{span } C$ bottom. To do so, we have used the matrices $A_0, P(-p, 0)_0, B_0$ and C_0 as defined in p. 5:

$$A_0 = \begin{bmatrix} 2 & 1 & -5 \\ -2 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad P(-p, 0)_0 = \begin{bmatrix} 0 & 3 & -5 \\ -4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 2 & -5 \\ -3 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

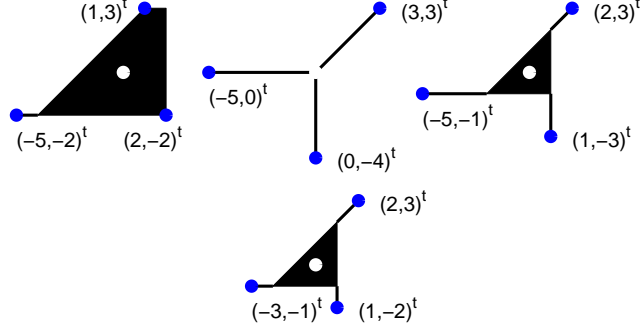


Figure 1: Top: $\{x_3 = 0\} \cap \text{span } A$ (left), $\{x_3 = 0\} \cap \text{span } P(-p, 0)$ (center) and $\{x_3 = 0\} \cap \text{span } B$ (right), for $p = (4, 3, 5)$. Bottom: $\{x_3 = 0\} \cap \text{span } C$, with $C = AB = BA$. In each case, the zero vector is marked in white and generators are represented in blue. The matrices A, B and C are perturbations of $P(-p, 0)$.

5 Geometry

Let $A, B \in M_n^{\text{nor}}$ be real. Here we study the role played by the geometry of the complexes $\text{span } A$ and $\text{span } B$ in order to have $AB = BA$. To do so, we bear in mind how the maps f_A and f_B act, where $f_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ transforms a column vector X into the product AX . For $n = 3$, f_A is described in detail in see [28]; see also [31].

Before, we have met two instances where the geometry explains why $AB = BA$. Namely, in remarks after propositions 7 and 8. In the first (resp. second) case we have $AB = BA = A$ (resp. $AB = BA = B$) because $\text{span } B$ is much larger (resp. smaller) than $\text{span } A$.

More generally, we explore the relationship among the sets $\text{span } A$, $\text{span } B$, $\text{span}(AB)$ and $\text{span}(BA)$ when commutativity is present or absent. In general, we have $\text{span}(AB) \subseteq \text{span } A$ and $\text{span}(BA) \subseteq \text{span } B$. In particular, if $AB = BA$ then $\text{span}(AB) \subseteq \text{span } A \cap \text{span } B$.

Proposition 24. *Let $A, B \in M_n^{\text{nor}}$. If $A \leq B = B^2$ and A is real, then $A \in \Omega^B(B)$ and $\text{span } A \supseteq \text{span } B$.*

Proof. By normality, we have $I \leq A \leq B \leq 0$ and left or right tropical multiplication by any matrix is monotonic. Therefore, $B \leq AB \leq B^2 = B$ and

$B \leq BA \leq B^2 = B$, whence $AB = BA = B$ and $A \in \Omega^B(B)$. Moreover, whatever the matrices A and B may be, we have $\text{span } A \supseteq \text{span}(AB)$ and, in our case, $\text{span}(AB) = \text{span } B$. \square

The hypothesis $B = B^2$ cannot be removed in the previous proposition, as the following example shows.

Example 25. Consider

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \leq B = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\text{span } A \not\subseteq \text{span } B$; see figure 2.

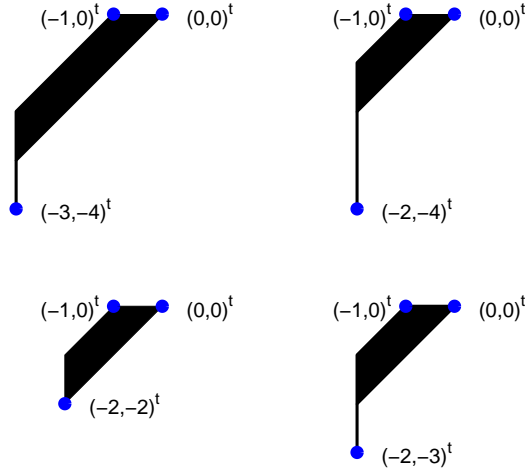


Figure 2: Top: $\{x_3 = 0\} \cap \text{span } A$ (left), $\{x_3 = 0\} \cap \text{span } B$ (right). Bottom: $\{x_3 = 0\} \cap \text{span}(AB)$ (left), $\{x_3 = 0\} \cap \text{span}(BA)$ (right). In this case, $\text{span } A \cap \text{span } B = \text{span}(AB)$. Generators are represented in blue.

Below we explore the properties of the matrices \underline{A} , A and \overline{A} and of the corresponding polyhedral complexes.

Example 11. (Continued) By proposition 17, we have

$$\underline{B} = \begin{bmatrix} 0 & -3 & -3 \\ -5 & 0 & -6 \\ -5 & -2 & 0 \end{bmatrix} \leq B = \begin{bmatrix} 0 & -3 & -1 \\ -4 & 0 & -6 \\ -5 & 0 & 0 \end{bmatrix} \leq \overline{B} = \begin{bmatrix} 0 & -1 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}$$

and we can easily check, in this case, that

$$\text{span } \underline{B} \supseteq \text{span } B \supseteq \text{span } \overline{B}.$$

See figure 3, where we are using the matrices

$$\underline{B}_0 = \begin{bmatrix} 5 & -1 & -3 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 5 & -3 & -1 \\ 1 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}, \overline{B}_0 = \begin{bmatrix} 4 & -1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix},$$

as defined in p. 5. Notice that $\{x_3 = 0\} \cap \text{span } B$ is the union of one closed 2-dimensional cell (called soma) and three closed 1-dimensional cells (called antennas); see [28] for the definition of soma, antennas and co-antennas (with a slightly different notation and language). In figure 3, bottom, we can see $\{x_3 = 0\} \cap \text{span } B$ together with its co-antennas.

In this example,

$$\overline{B} = B^*$$

and the matrix \underline{B} is idempotent. Therefore, the sets $\text{span } \underline{B}$ and $\text{span } \overline{B}$ are classically convex, and so are the sections $\{x_3 = 0\} \cap \text{span } \underline{B}$ and $\{x_3 = 0\} \cap \text{span } \overline{B}$.

Consider \mathcal{H} , the classical convex hull of $\{x_3 = 0\} \cap \text{span } B$: its the vertices are $(5, 0)^t$, $(5, 1)^t$, $(-2, 1)^t$, $(-3, 0)^t$, $(-3, -6)^t$ and $(-1, -6)^t$, going counterclockwise. Notice that $\{x_3 = 0\} \cap \text{span } \underline{B}$ is strictly larger than \mathcal{H} . Actually, $\{x_3 = 0\} \cap \text{span } \underline{B}$ is the convex hull of the union of $\{x_3 = 0\} \cap \text{span } B$ and the co-antennas of it. On the other hand, $\{x_3 = 0\} \cap \text{span } \overline{B}$ is the soma of $\{x_3 = 0\} \cap \text{span } B$, i.e., it is the maximal convex set contained there. \square

We wonder whether the statements in the former example are true for any real $B \in M_n^{\text{nor}}$. This is an open QUESTION.

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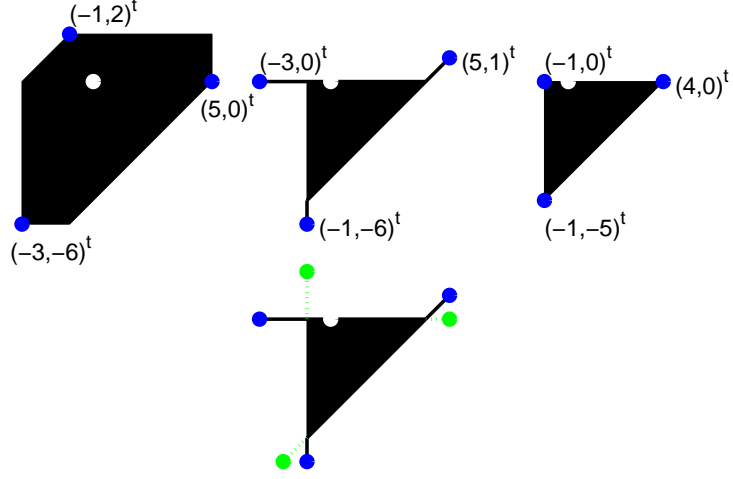


Figure 3: Top: $\{x_3 = 0\} \cap \text{span } \underline{B}$ (left), $\{x_3 = 0\} \cap \text{span } B$ (center) and $\{x_3 = 0\} \cap \text{span } \overline{B}$ (right), for B in (21). In each case, the zero vector is marked in white, and generators (i.e., columns of the corresponding matrix \underline{B}_0 , B_0 and \overline{B}_0) are represented in blue. The hyperplane section $\{x_3 = 0\} \cap \text{span } B$ has three antennas. Bottom: $\{x_3 = 0\} \cap \text{span } B$ is represented together with its co-antennas, which appear dotted in green. The convex hull of the bottom figure is the top left one.

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